RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College under University of Calcutta)

B.A./B.SC. FIRST SEMESTER EXAMINATION, DECEMBER 2013

FIRST YEAR

Date : 14/12/2013 **Mathematics (Honours)**

Time : 11am – 3pm Paper : I Full Marks : 100

(Use separate answer book for each group)

Group – A

Question No. 1 is compulsory. Answer **any two** from question No. 2 - 4 and **any two** from question No. 5 - 7. Symbols have their usual meanings.

1. State whether the following statements one true or false. Justify your answer.
   a) If for 3 sets A, B, C we have, $A \Delta B = A \Delta C$, then B = C.
   b) If $f : A \to B$ be a map and $P \subseteq A$, then $f^{-1} f(P) = P$.
   c) $\mathbb{Z}_m$ is a subgroup of $\mathbb{Z}_n$ if m divides n.
   d) If every proper subgroup of a group is cyclic, then the group itself is cyclic.
   e) The order of A_n is $\frac{1}{2} n!$ (5)
2. Show that the set G of all ordered pairs (a, b) with $a \neq 0$ of real numbers a, b is a group with operation $*$ defined by $(a, b) * (c, d) = (ac, bc + d)$. (5)
3. Let H and K be subgroups of a group G. show that HK is a subgroup of G if and only if HK=KH. (5)
4. State lagrange is theorem (on the order of a subgroup of a finite group). Prove its converse in case of a finite cyclic group. (1 + 4)
5. i) Determine whether the given permutation
   $$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 7 & 9 & 1 & 8 & 2 & 6 & 3 & 5 \end{pmatrix}$$
   is an even permulation. (2)
   ii) let H be a subgroup of a group G. A relation ρ on the set G is defined by
   $$\rho = \{(a, b) \in G \times G \mid a^{-1} b \in H\}$$
   Prove that ρ is an equivalence relation on G. (3)
6. Let $\phi : P(\mathbb{R}) \to P(\mathbb{R})$ be defined by $\phi(A) = \bar{A}$ (the closure of A). check if ϕ is
   i) one - one,
   ii) onto. (2 + 3)
7. Prove that for $n \geq 3$, the subgroup of S_n generated by the 3 - cycles is A_n. (5)

Answer **any five** questions of the following : (5 × 5)

8. a) Define infimum of a subset A of R where A is bounded below. Let $q \in R - Q$. Does there exist a set $A \subseteq Q$ such that $\inf A = q$? Justify your answer. (1 + 2)
   b) Prove that 'o' is an infimum of infinitely many subsets of R. (2)
9. Show that the intersection of a finite number of open sets in $\mathbb{R}$ is an open set.
   With the help of a suitable example show that the intersection of an infinite collection of open sets need not always be an open set. (3 +2)
10. Prove that the set $R - Q$ is uncountable. (5)
11. a) If A is an uncountable subset of R, show that A has a limit point. (3)
    b) Give an example of a subset A of R such that $A^d = \{0,1\}$, A^d denotes the derived set of A. (2)

12. Let $\{I_n\}_n$ be a sequence of nonempty bounded closed intervals such that $I_{n+1} \subseteq I_n$ for all n. show that there exists at least one point ξ such that $\xi \in \bigcap_{n=1}^{\alpha} I_n$.

Is the conclusion true if $\{I_n\}_n$ be a sequence of open intervals? Justify your answer. (4 + 1)

13. a) Prove that every bounded sequence of real numbers has a convergent subsequence. (3)

b) Give an example of a divergent sequence having only one sub sequential limit. (2)

14. Prove that a bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ is convergent if and only if

$\overline{\lim}\, x_n = \underline{\lim}\, x_n$. (5)

15. a) Let a montone increasing function f be bounded below on the bounded open interval (a, b). show that $\lim_{x\to a+} f(x) = \inf_{x\in(a,b)} f(x)$. (3)

b) let $f, g : D \to \mathbb{R}$, where $DC\mathbb{R}$, be two functions. Let p be and accumulation point of D. if $\lim_{n\to p} f(x) = 0$ and g be bounded on D, then show that $\lim_{n\to p} f(x)g(x) = 0$. (2)

Group – B

Answer **any five** questions from questions (16 - 23) and **any five** from questions (24 - 31) : (10 × 5)

16. If the lines $ax^2 + 2hxy + by^2 = 0$ be two sides of a parallelogram and the line $\ell x + my = 1$ be one of its diagonals, show that the equation of the other diagonal is $(am - h\ell)x = (b\ell - hm)y$. (5)

17. Any tangent to an ellipse with centre C meets the director circle in P and D. Prove that CP and CD are in the directions of the conjugate diameters of the ellipse. (5)

18. Prove that the conics $\frac{\ell_1}{r} = 1 - e_1 \cos\theta$ and $\frac{\ell_2}{r} = 1 - e_2 \cos(\theta - \alpha)$ will touch one another if $\ell_1{}^2(1 - e_2{}^2) + \ell_2{}^2(1 - e_1{}^2) = 2\ell_1\ell_2(1 - e_1e_2\cos\alpha)$. (5)

19. Reducing the equation $4x^2 + 4xy + y^2 - 4x - 2y + a = 0$ to its cannoical form, determine the nature of the conic for different values of a. (5)

20. Prove that the locus of the poles of normal chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with respect to the hyperbola is the curve $a^6y^2 - b^6x^2 = (a^2 + b^2)^2 x^2 y^2$. (5)

21. a) Find the equation of the straight line passing through a point whose position vector is $\vec{b}$ and which is perpendicular to $\vec{r} = \vec{a} + t\vec{c}$. (3)

b) A force of 15 units acts in the direction of the vector $\vec{a} = (1, -2, 2)$ and passes through the point (2, -2, 2). Find the moment of the force about the point (1, 1, 1). (2)

22. Show by vector method that the vector area of the triangle formed by joining mid point of one of the non-parallel sides of a trapezium to the extremities to the opposite side is half of the vector area of the trapezium. (5)

23. Prove the formula:

$(\vec{b}\times\vec{c}).(\vec{a}\times\vec{d}) + (\vec{c}\times\vec{a}).(\vec{b}\times\vec{d}) + (\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = 0$

and use it to show that

$\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$

for any two acute angles A and B. (2 + 3)

24. Show that the substitution $z = ax + by + c$ changes $\frac{dy}{dx} = f(ax + by + c)$ into an equation with separable variables, and apply this method to solve the following equation : $\frac{dy}{dx} = \sin^2(x - y + 1)$. (2 + 3)

25. a) Find the orthogonal trajectories of the family $y = x + ce^{-x}$. (3)

b) Under what circumstances will the differential equation $M(x, y)dx + N(x, y)dy = 0$ have an integrating factor that is a function of the sum $z = x + y$? (2)

26. Verify that the equation $(2x^2 + 3x)\frac{d^2y}{dx^2} + (6x + 3)\frac{dy}{dx} + 2y = (x + 1)e^x$ is exact and then solve it. (5)

27. Solve $x^2\frac{d^2y}{dx^2} - x(x + 2)\frac{dy}{dx} + (x + 2)y = x^3$ given that y = x, y = xe^x are two linearly independents solutions of the corresponding homogeneous equation by the method of variation of parameters. (5)

28. Solve : $(x + a)^2\frac{d^2y}{dx^2} - 4(x + a)\frac{dy}{dx} + 6y = x$. (5)

29. Solve by the method of undetermined coefficients : $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3y = x^3 + \sin x$. (5)

30. Solve the differential equation $\frac{d^2y}{dx^2} + a^2y = \sec ax$ with the symbolic operator D. (5)

31. Reduce the equation $\sin y\frac{dy}{dx} = \cos x(2\cos y - \sin^2 x)$ to a linear equation and hence solve it. (1 + 4)